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The Study of Short Account of Wronskian and Eigen values with matrix form

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Abstract:

In the present paper we study the Wronskian and existence of eigenvalue problems in the case of ordinary linear differential equations with different examples and lemmas associated with matrix method, including certain boundary conditions. By using a necessary and sufficient condition that a set of 2n solutions with some basis of fundamental set of solutions. The numerical results are discussed in the illustrated examples. The important results are also discussed in the lemmas in this paper. The purpose of the study is to more clearly clarify the concept to apply Wronskian and eigenvalues in different cases.

1. Introduction

Let the matrix equation

$$(L-\lambda)\phi = 0 \qquad (2.1.3)$$

where λ is a parameter, real or complex. The Wronskian for the system (2.1.3) we consider

$$\psi_j \equiv \psi_j(x,\lambda) = \begin{bmatrix} u_{j_1} \\ u_{j_2} \\ \vdots \\ u_{j_n} \end{bmatrix} \equiv \begin{bmatrix} u_{j_1}(x,\lambda) \\ u_{j_2}(x,\lambda) \\ \vdots \\ u_{j_n}(x,\lambda) \end{bmatrix} \quad j = 1,2,3....2n$$

be 2n-solutions of (2.1.3). Then the determinant

$$W_x(\psi_1,\psi_2......\psi_{2n})(\lambda) \equiv W\{\psi_1(x,\lambda),\psi_2(x,\lambda),...,\psi_{2n}(x,\lambda)\}$$

defined by

$$W_x(\psi_1,\psi_2.....,\psi_{2n})\lambda = \begin{vmatrix} u_{11} & u_{21}..........u_{2n1} \\ u_{12} & u_{22}..........u_{2n2} \\ \multicolumn{2}{l}{.............................} \\ u_{1n} & u_{2n}...........u_{2n,n} \\ u'_{11} & u'_{21}...........u'_{2n,1} \\ u'_{12} & u'_{22}...........u'_{2n,2} \\ \multicolumn{2}{l}{..............................} \\ u'_{1n} & u'_{2n}............u'_{2nn} \end{vmatrix} \qquad (2.1.4)$$

is called the Wronskian for the system (2.1.3) as it plays the same role for the system (2.1.3) as does the Wronskian in the case of ordinary linear differential equation. Bhagat [11] and Levinson [12] have studied the necessary and sufficient condition that the 2n– solutions $\psi_1, \psi_2 .....\psi_{2n}$ of (2.1.3) linearly independent is that $W_x(\psi_1, \psi_2.....\psi_{2n})(\lambda) \neq 0$ from the following

Theorem

Consider

$$F \equiv F(x) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \text{ and } G \equiv G(x) \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

where f_1, f_2,…….f_n; g_1, g_2…..,g_n are functions of a be two vectors having continuous derivatives of second order. Then from the result

$$\phi_K^T L\phi j - (L\phi_k)^T \phi_j = \left[\phi_j, \phi_k\right]'$$ where ϕ_j and ϕ_k are the solutions of (2.1.3) we have

$$G^T LF - F^T LG - [F\ G]'$$

Thus

$$\int_a^b (G^T LF - F^T LG)dx = \left[F\ G\right]_a^b = \left[F\ \ G\right](b) - \left[F\ \ G\right](a) \qquad (2.1.5)$$

This is the Green's Formula for our boundary value problem

Now if F and G are such that [F G] (b) – [F G] (a) = 0

Then

$$\int_a^b G^T LF dx = \int_a^b F^T LG\, dx \qquad (2.1.6)$$

the self adjointness condition

Let F and G be the solutions of (2.1.3) and satisfy the boundary conditions

$$aj_1\phi_1 + aj_2\phi_1^1 + aj_3\phi_2 + aj_4\phi_2^1 + ... + aj_{2n-1}\phi_n + aj_{2n}\phi_n^1 = 0, j = 1, 2....n$$

and $bj_1\phi_1 + bj_2\phi_1^1 + bj_3\phi_2 + bj_4\phi_2^1 + ... + bj_{2n-1}\phi_n + bj_{2n}\phi_n^1 = 0$

j = 1, 2,…….n

at x = a and x = b respectively. If the conditions are satisfied then by the theorem

$$\left[F\ G\right](b) = [F\ G](a) = 0$$

Let

$$\psi_j(x, \lambda) = \begin{bmatrix} uj_1(x, \lambda) \\ uj_2(x, \lambda) \\ \vdots \\ uj_n(x, \lambda) \end{bmatrix} j = 1, 2, 3......2n$$

be 2n. solution of (2.1.3) for the same values of λ. Then the Wronskian $W_x(\psi_1,\psi_2,......,\psi_n)(\lambda)$ is independent of x and depends only on λ.

Let

$$A=\begin{bmatrix} u'_{11} & u'_{12}........u'_{1n} & u_{11} & u_{12}........u_{1n} \\ u'_{21} & u'_{22}........u'_{2n} & u_{21} & u_{22}........u_{2n} \\ \multicolumn{4}{c}{..} \\ u'_{2n1} & u'_{2n2}......u'_{2n.n} & u_{2n1} & u_{2n2}........u_{2nn} \end{bmatrix}$$

$$B=\begin{bmatrix} u_{11} & u_{12}........u_{1n} & -u'_{11} & -u'_{12}........-u'_{1n} \\ u_{21} & u_{22}........u_{2n} & -u'_{21} & -u'_{22}........-u'_{2n} \\ \multicolumn{4}{c}{..} \\ u_{2n1} & u_{2n2}......u_{2nn} & -u'_{2n1} & -u'_{2n2}..........-u'_{2nn} \end{bmatrix}$$

Then we have

$$\left[[\psi_r\psi_s],1\le r;s\le 2n\right]=\begin{bmatrix} [\psi_1\ \ \psi_1]\ [\psi_1\ \ \psi_2]......[\psi_1\ \ \psi_{2n}] \\ [\psi_2\ \ \psi_1]\ [\psi_2\ \ \psi_2]......[\psi_2\ \ \psi_{2n}] \\ .... \\ [\psi_{2n}\ \ \psi_1]\ [\psi_{2n}\ \ \psi_2].....[\psi_{2n}\ \psi_{2n}] \end{bmatrix}=AB^T$$

Hence

$$\det_{1\le r,s\le 2n}\left[[\psi_r\psi_s]\right]=\det A.\det B$$

But $\det A=(-1)^{n^2}W_x(\psi_1,\psi_2,.....,\psi_{2n})(\lambda)$

and $\det B=(-1)^{n}W_x(\psi_1,\psi_2,.....,\psi_{2n})(\lambda)$

Hence,

$$1\le r,s\le 2n\left[[\psi_r\psi_s]\right]=(-1)^{n(n+1)}\left\{W_x(\psi_1,\psi_2.....\psi_{2n})(\lambda)\right\}^2$$

i.e. $$\det_{1\le r,s\le 2n}\left[[\psi_r\psi_s]\right]=\left\{W_x(\psi_1,\psi_2,.....,\psi_{2n})(\lambda)\right\}^2 \qquad (2.1.7)$$

It follows from (2.1.7) that $W_x(\psi_1,\psi_2,.....,\psi_{2n})(\lambda)$ is independent of x and depends only on λ since each $[\psi_r\psi_s]$ is so.

If ϕ_1, ϕ_1…… ϕ_{2n} are boundary condition vectors defined in $[\phi_i\ \phi_j]=0$ $1\le i,\ j\le n$ and $[\phi_k\phi_\ell]=0\quad n+1\le k,\ \ell\le 2n$ and (2.1.7) we have

$$\left\{W_x(\phi_1,\phi_2,........,\phi_{2n}(\lambda)\right\}^2$$

$$\begin{vmatrix} [\phi_1\ \phi_{n+1}]\,[\phi_1\,\phi_{n+2}]......[\phi_1\,\phi_{2n}] \\ [\phi_2\ \phi_{n+1}]\,[\phi_2\,\phi_{n+2}]......[\phi_2\,\phi_{2n}] \\ .... \\ [\phi_n\ \phi_{n+1}]\,[\phi_n\,\phi_{n+2}]......[\phi_n\,\phi_{2n}] \end{vmatrix}^2 (\lambda)$$

we define

$$D(\lambda) = \begin{vmatrix} [\phi_1\ \phi_{n+1}]\,[\phi_1\,\phi_{n+2}]......[\phi_1\,\phi_{2n}] \\ [\phi_2\ \phi_{n+1}]\,[\phi_2\,\phi_{n+2}]......[\phi_2\,\phi_{2n}] \\ .... \\ [\phi_n\ \phi_{n+1}]\,[\phi_n\,\phi_{n+2}]......[\phi_n\,\phi_{2n}] \end{vmatrix} (\lambda) \qquad (2.1.8)$$

Hence

$$\left\{W_x\left(\phi_1,\phi_2,......,\phi_{2n}\right)(\lambda)\right\}^2 = \{D(\lambda)\}^2 \qquad (2.1.9)$$

Here D (λ) is an integral function of λ independent of x and real for real λ. Hence in the complex λ-plane the zeros of D(λ) form an isolated set whose only limit point is at infinity. Consequently the zeros of D(λ) form an almost enumerable set. Hence D(λ) is not identically zero in x over [a, b] and consequently $W_x(\phi_1,\phi_2......,\phi_{2n})(\lambda)$ is not identically zero in x over [a. b]. Hence the boundary condition vectors $\phi_1,\phi_2......,\phi_{2n}$ are linearly independent over [a, b] and form a fundamental set for those values of λ for which $D(\lambda)\neq 0$. If $D(\lambda)=0$ for some λ, then ϕ_1, ϕ_2 …. ϕ_n are linearly dependent.

We now prove a theorem on the existence of eigen values for this, consider the necessary and sufficient condition that λ should be an eigen value is that it is a root of D(λ) = 0

Suppose that $\lambda = \lambda_1$, $D(\lambda)\neq 0$ i.e. $W(\lambda)\neq 0$

Then $\phi_1,\phi_2,......,\phi_{2n}$ form a fundamental set for the differential system

$$\frac{d^2\phi_1}{dx^2} - A_{11}\phi_1 - A_{12}\phi_2...... - A_{1n}\phi_n = \lambda\left(\phi_1+\phi_2+...+\phi_n\right)$$

$$\frac{d^2\phi_2}{dx^2} - A_{21}\phi_1 - A_{22}\phi_2...... - A_{2n}\phi_n = \lambda\left(\phi_1+\phi_2+...+\phi_n\right)$$

……………………………………………………………

$$\frac{d^2\phi_n}{dx^2} - A_{n1}\phi_1 - A_{n2}\phi_2...... - A_{nn}\ \phi_n = \lambda(\phi_1+\phi_2....+\phi_n) \qquad (2.1.10)$$

and any solution

$$\phi(x,\lambda_1)=\begin{bmatrix} u_1(x,\lambda_1) \\ u_2(x,\lambda_1) \\ \vdots \\ u_n(x,\lambda_1) \end{bmatrix}$$

of above equation can be expressed as $\phi(x,\lambda_1)=\sum_{j=1}^{2n}\alpha_j\phi_j(x,\lambda_1)$ where $\alpha_1,\alpha_2......\alpha_{2n}$ are constants (real or complex) not all zero.

Let $\phi(x,\lambda_1)$ satisfy the boundary conditions

$$\left[\theta(x,\lambda)\ \phi_j\left(\frac{a}{x},\lambda\right)\right]=0\quad j=1,2......n\ \text{ and}$$

$$\left[\theta(x,\lambda)\ \phi_k\left(\frac{b}{x},\lambda\right)\right]=0\quad k=n+1,n+2,......2n$$

Then since $\left[........\phi_k\left(x,\lambda_1\right)\right]$ is a linear operator

So,

$$\sum_{j=1}^{2n}\alpha_j\left[\phi_j\ \phi_k\right]=0,\quad k=1,2......2n \tag{2.1.11}$$

The necessary and sufficient condition that (2.1.11) have a non trivial solution is that the determinant of the coefficients should vanish. i.e.

$$\det_{1\le r,s\le 2n}\left[\left[\phi r,\phi_s\right]\right]\left(\lambda_1\right)=0$$

i.e. $D\left(\lambda_1\right)=0$ by (2.1.7) and (2.1.9)

But $D\left(\lambda_1\right)\neq 0.$ Hence the only solution of (2.1.10) is a trivial one. Therefore for $\lambda = \lambda_1$ no solution of (2.1.11)satisfying the boundary conditions

$$\left[\theta(x,\lambda)\ \phi_j\left(\frac{a}{x},\lambda\right)\right]=0,\quad j=1,2,3.....n\quad \text{and} \tag{2.1.12}$$

$$\left[\theta(x,\lambda)\ \phi_k\left(\frac{b}{x},\lambda\right)\right]=0,\quad k=n+1,n+2,....2n \tag{2.1.13}$$

Consequently $\lambda = \lambda_1$ is not an eigenvalue.

Conversely, Let λ_1 be a root of $D(\lambda)=0$

Then $W_x(\phi_1,\phi_2,....,\phi_{2n})(\lambda_1)$ is identically zero in x over [a, b]. Hence there exists a linear relation of the form

$$A_1\phi_1(x,\lambda_1)+A_2\phi_2(x,\lambda_1)+.....+A_n\phi_n(x,\lambda_1)$$

$$= A_{n+1}\phi_{n+1}(x,\lambda_1) + A_{n+2}\phi_{n+2}(x,\lambda_1) + ... + A_{2n}\phi_{2n}(x,\lambda_1) \qquad (2.1.14)$$

For all $x \in [a,b]$ where A_1, A_2,….,A_{2n} are constants with an important results are such that not all A_1, A_2……A_n are zero and not all of A_{n+1}, A_{n+2}……..,A_{2n} are zero.

Then the vector,

$$\phi(x,\lambda_1) = A_1\phi_1(x,\lambda_1) + A_2\phi_2(x,\lambda_2) + ... + A_n\phi_n(x_1,\lambda_1) \qquad (2.1.15)$$

Satisfies the system (2.1.10) and (2.1.14) it also satisfies the boundary conditions (2.1.12) and (2.1.13). Hence $\phi(x,\lambda_1)$ given by the above equation is an eigenvector and λ_1 is an eigenvalue.

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